

# ON VOJTA'S $1 + \varepsilon$ CONJECTURE

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## 1. INTRODUCTION

In [V1] and [V2], P. Vojta conjectured that

**Conjecture 1.1** ( $1 + \varepsilon$  Conjecture). *Let  $\pi : X \rightarrow B$  be a flat family of projective curves over a projective curve  $B$  with connected fibers. Suppose that  $X$  has at worst quotient singularities. Then for every  $\varepsilon > 0$ , there exists a constant  $N_\varepsilon$  such that*

$$(1.1) \quad \omega_{X/B} \cdot C \leq (1 + \varepsilon)(2g(C) - 2) + N_\varepsilon(X_b \cdot C)$$

for every irreducible curve  $C \subset X$  that dominates  $B$ , where  $\omega_{X/B}$  is the relative dualizing sheaf of  $X/B$ ,  $X_b$  is a general fiber of  $X/B$  and  $g(C)$  is the geometric genus of  $C$ .

*Remark 1.2.* From the number-theoretical point of view, one can think of  $X$  as an algebraic curve  $X_k$  over the function field  $k = K(B)$  and the multi-section  $C \subset X$  as an algebraic point  $p_C$  on  $X_{\bar{k}} = X_k \otimes \bar{k}$ . The logarithmic height  $h(p_C)$  and discriminant  $\Delta(p_C)$  of  $p_C$  are defined to be

$$(1.2) \quad h(p_C) = \frac{\omega_{X/B} \cdot C}{\deg(K(C)/K(B))} \text{ and } \Delta(p_C) = \frac{2g(C) - 2}{\deg(K(C)/K(B))}$$

respectively, where  $\deg(K(C)/K(B)) = X_b \cdot C$ , obviously. With these notations, (1.1) can be put in the form

$$(1.3) \quad h(p_C) \leq (1 + \varepsilon)\Delta(p_C) + N_\varepsilon$$

Note that the definition of the height  $h(p_C)$  depends on the choice of the birational model  $X$  of  $X_k$ . However, it is not hard to see that (1.3) holds regardless of the choice of the birational model (see below).

Vojta proved that (1.1) holds with  $1 + \varepsilon$  replaced by  $2 + \varepsilon$ . This conjecture was settled recently by K. Yamanoi [Y]. M. McQuillan later gave an algebro-geometric proof in [M]. However, we find his proof quite hard to follow. Inspired by his idea, we will give another proof of this conjecture and generalize it to the log case. Compared to his proof, ours is more elementary.

It turns out natural to study a (generalized) log version of the  $1 + \varepsilon$  conjecture. For a log variety  $(X, D)$  and a curve  $C \subset X$  that meets  $D$  properly, we define  $i_X(C, D)$  to be the number of the points in  $\nu^{-1}(D)$ , where  $\nu : \tilde{C} \rightarrow C \subset X$  is the normalization of  $C$ .

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*Date:* Dec 7, 2006.

**Conjecture 1.3.** *Let  $\pi : X \rightarrow B$  be a flat family of projective curves over a projective curve  $B$  with connected fibers. Suppose that  $X$  has at worst quotient singularities and  $D \subset X$  is a reduced effective divisor on  $X$ . Then for every  $\varepsilon > 0$ , there exists a constant  $N_\varepsilon$  such that*

$$(1.4) \quad (\omega_{X/B} + D) \cdot C \leq (1 + \varepsilon)(2g(C) - 2 + i_X(C, D)) + N_\varepsilon(X_b \cdot C)$$

for every irreducible curve  $C \subset X$  that dominates  $B$  and  $C \not\subset D$ .

## 2. REDUCTION TO $(\mathbb{P}^1 \times B, D)$

As a first step in our proof, we will reduce Conjecture 1.3 to the case  $(\mathbb{P}^1 \times B, D)$ . This is also what was done in Yamanoi's proof [Y].

It is not hard to see that (1.4) continues to hold after applying birational transforms and/or base changes to  $X/B$ . That is, we have

**Lemma 2.1.** *Let  $\pi : X \rightarrow B$  and  $D$  be given as in Conjecture 1.3.*

- (1) *Let  $f : X' \dashrightarrow X$  be a birational morphism and  $D'$  be the proper transform of  $D$  under  $f$ . Then (1.4) holds for  $(X, D)$  if and only if it holds for  $(X', D')$ .*
- (2) *Let  $B' \rightarrow B$  be a finite map from a smooth projective curve  $B'$  to  $B$ ,  $f : X' = X \times_B B' \rightarrow X$  be the base change of the family  $X$  and  $D' = f^{-1}(D)$ . Then (1.4) holds for  $(X, D)$  if and only if it holds for  $(X', D')$ .*

The constants  $N'_\varepsilon$  for  $(X', D')$ , though, might be different from  $N_\varepsilon$  for  $(X, D)$ .

*Proof.* For part (1), it is enough to argue for  $X'$  being the blowup of  $X$  at one point  $p$ . Let  $C' \subset X'$  be the proper transform of  $C \subset X$ . Then

$$(2.1) \quad (\omega_{X/B} + D) \cdot C = (\omega_{X'/B} + D' + rE) \cdot C'$$

for some constant  $r$ , where  $E$  is the exceptional divisor of  $f$ . On the other hand, we have

$$(2.2) \quad E \cdot C' \leq X'_b \cdot C' = X_b \cdot C = \deg(C)$$

where  $X'_b$  and  $X_b$  are the fibers of  $X'$  and  $X$  over a point  $b \in B$ , respectively. Consequently,

$$(2.3) \quad |(\omega_{X/B} + D) \cdot C - (\omega_{X'/B} + D') \cdot C'| \leq |r| \deg C$$

Also it is obvious that  $g(C) = g(C')$  and

$$(2.4) \quad |i_X(C, D) - i_{X'}(C', D')| \leq E \cdot C' \leq \deg(C)$$

Then part (1) follows from (2.3) and (2.4).

For part (2), let  $d$  be the degree of the map  $B' \rightarrow B$ ,  $R \subset B'$  be its ramification locus and  $\mu_r$  be the ramification index of a point  $r \in R$ . Let  $C' = f^*(C)$ . It is not hard to see that

$$(2.5) \quad |d(\omega_{X/B} + D) \cdot C - (\omega_{X'/B'} + D') \cdot C'| \leq \sum_{r \in R} (\mu_r - 1) \deg(C)$$

$$(2.6) \quad |d(2g(C) - 2) - (2g(C') - 2)| \leq \sum_{r \in R} (\mu_r - 1) \deg C$$

and

$$(2.7) \quad |d(i_X(C, D)) - i_{X'}(C', D')| \leq \sum_{r \in R} (\mu_r - 1) \deg C$$

Then part (2) follows from (2.5)-(2.7).  $\square$

*Remark 2.2.* We see from the above lemma that (1.3) holds regardless of the choice of birational models  $X$ .

*Remark 2.3.* If  $(\omega_{X/B} + D) \cdot X_b \leq 0$ , (1.4) is trivially true. So we may assume that

$$(2.8) \quad (\omega_{X/B} + D) \cdot X_b > 0.$$

We may also assume that  $D$  meets every fiber properly. Using the above lemma, we can apply the stable reduction to  $(X, D)$  and make  $X$  into a family of stable curves with marked points  $X_b \cap D$  on each fiber. The resulting  $X$  has at worst quotient singularities and  $\omega_{X/B} + D$  is relatively ample over  $B$ .

**Proposition 2.4.** *If (1.4) fails for some  $(X, D)$ , then there exists  $\delta > 0$  and a log pair  $(Y, R)$  such that (1.4) fails with  $(X, D, \varepsilon)$  replaced by  $(Y, R, \delta)$ , where  $R$  is a reduced effective divisor on  $Y = \mathbb{P}^1 \times B$ .*

*Proof.* By the above remark, we may assume that  $X$  is a family of stable curves with marked points  $X_b \cap D$ . In particular,  $\omega_{X/B} + D$  is relatively ample over  $B$ .

Since (1.4) fails for  $(X, D)$ , there exists a sequence of irreducible curves  $C_1, C_2, \dots, C_n, \dots \subset X$  such that

$$(2.9) \quad \lim_{n \rightarrow \infty} \left( \frac{(\omega_{X/B} + D) \cdot C_n}{X_b \cdot C_n} - \frac{(1 + \varepsilon)(2g(C_n) - 2 + i_X(C_n, D))}{X_b \cdot C_n} \right) = \infty$$

Taking a sufficiently ample line bundle  $L$  on  $X$ , we can map  $X \rightarrow \mathbb{P}^1$  with a very general pencil in  $|L|$ . Combining this with the projection  $X \rightarrow B$ , we obtain a rational map  $\phi : X \dashrightarrow Y = B \times \mathbb{P}^1$ . We can make the following happen by taking  $L$  sufficiently ample and the pencil sufficiently general:

- The indeterminacy locus  $I_\phi$  of  $\phi$  consists of  $L^2$  distinct points on  $X$ ,  $I_\phi \cap C_n = \emptyset$  for all  $n$  and  $I_\phi \cap D = \emptyset$ .
- Outside of  $I_\phi$ ,  $\phi$  is finite. Let  $R_X \subset X$  be the closure of the ramification locus of  $\phi : X \setminus I_\phi \rightarrow Y$ ,  $R_Y = \overline{\phi(R_X)}$  be the proper transform of  $R_X$  and

$$(2.10) \quad \phi^* R_Y = 2R_X + R_\phi$$

outside of  $I_\phi$ , where  $R_\phi \subset X$  is a reduced effective divisor on  $X$ .

- $\phi$  is simply ramified along  $R_X$  with multiplicity 2.
- $\phi$  maps  $C_n$  and  $D$  birationally to  $\Gamma_n = \phi(C_n)$  and  $\Delta = \phi(D)$ , respectively, for all  $n$ .

Since  $\phi_* C_n = \Gamma_n$ , we have

$$(2.11) \quad \phi^*(\omega_{Y/B} + R_Y + \Delta) \cdot C_n = (\omega_{Y/B} + R_Y + \Delta) \cdot \Gamma_n$$

On the other hand,

$$(2.12) \quad \begin{aligned} & \phi^*(\omega_{Y/B} + R_Y + \Delta) \cdot C_n \\ &= (\phi^*\omega_{Y/B} + 2R_X + R_\phi + \phi^*\Delta) \cdot C_n \\ &= (\phi^*\omega_{Y/B} + R_X + D) \cdot C_n + (R_X + R_\phi) \cdot C_n + D_\phi \cdot C_n \end{aligned}$$

where

$$(2.13) \quad \phi^*\Delta = D + D_\phi$$

for some effective divisor  $D_\phi \subset X$ . By Riemann-Hurwitz,

$$(2.14) \quad \omega_{X/B} = \phi^*\omega_{Y/B} + R_X$$

holds outside of  $I_\phi$ . Meanwhile, it is obvious that

$$(2.15) \quad (R_X + R_\phi) \cdot C_n \geq i_Y(\Gamma_n, R_Y)$$

and

$$(2.16) \quad D_\phi \cdot C_n \geq i_Y(\Gamma_n, \Delta) - i_X(C_n, D)$$

Combining (2.11) through (2.16), we obtain

$$(2.17) \quad \begin{aligned} & (\omega_{Y/B} + R_Y + \Delta) \cdot \Gamma_n - (1 + \delta)(2g(\Gamma_n) - 2 + i_Y(\Gamma_n, R)) \\ & \geq (\omega_{X/B} + D) \cdot C_n - (1 + \delta)(2g(C_n) - 2 + i_X(C_n, D)) \\ & \quad - \delta(R_X + R_\phi + D_\phi)C_n \end{aligned}$$

where  $R = R_Y \cup \Delta$ . Since  $\omega_{X/B} + D$  is relatively ample over  $B$ , there exist constants  $\beta$  and  $\gamma > 0$  such that

$$(2.18) \quad (R_X + R_\phi + D_\phi)C \leq \gamma(\omega_{X/B} + D + \beta X_b)C$$

for all curves  $C \subset Y$ . Thus, it suffices to take

$$(2.19) \quad \delta = \frac{\varepsilon}{(1 + \varepsilon)\gamma + 1}$$

Then

$$(2.20) \quad \begin{aligned} & (\omega_{X/B} + D) \cdot C_n - (1 + \delta)(2g(C_n) - 2 + i_X(C_n, D)) \\ & \quad - \delta(R_X + R_\phi + D_\phi) \cdot C_n \\ & \geq (1 - \delta\gamma)(\omega_{X/B} + D) \cdot C_n - (1 + \delta)(2g(C_n) - 2 + i_X(C_n, D)) \\ & \quad - \beta\gamma\delta(X_b \cdot C_n) \\ & = (1 - \delta\gamma) \left( (\omega_{X/B} + D) \cdot C_n - (1 + \varepsilon)(2g(C_n) - 2 + i_X(C_n, D)) \right) \\ & \quad - \beta\gamma\delta(X_b \cdot C_n) \end{aligned}$$

Therefore,

$$(2.21) \quad \lim_{n \rightarrow \infty} \left( \frac{(\omega_{Y/B} + R) \cdot \Gamma_n}{Y_b \cdot \Gamma_n} - (1 + \delta) \frac{2g(\Gamma_n) - 2 + i_Y(\Gamma_n, R)}{Y_b \cdot \Gamma_n} \right) = \infty$$

and Proposition 2.4 follows.  $\square$

In the above proof, we have quite an amount of freedom to choose the map  $X \dashrightarrow \mathbb{P}^1$ . We can make  $R$  really “nice” by choosing  $L$  and the pencil of  $L$  sufficiently “general”.

**Proposition 2.5.** *Let  $S$  be a finite set of points on  $B$ . In the proof of Proposition 2.4, for a sufficiently ample  $L$  and a general pencil  $\sigma \subset |L|$  that maps  $X \dashrightarrow \mathbb{P}^1$ , the corresponding divisor  $R = R_Y + \Delta \subset Y = \mathbb{P}^1 \times B$  has the following properties:*

- For every fiber  $Y_b$  of  $Y/B$ ,

$$(2.22) \quad i_Y(Y_b, R) \geq Y_b \cdot R - 1$$

and if the equality holds,  $b \in B \setminus S$  and  $X_b$  is disjoint from the base locus  $\text{Bs}(\sigma)$  of  $\sigma$ ;

- $R$  is a divisor of normal crossing.

*Proof.* Let  $\mathbb{G}(k, |L|)$  be the Grassmanian  $\{\mathbb{P}^k \subset |L|\}$ . For each pencil  $\sigma \in \mathbb{G}(1, |L|)$ , we use the notation  $\phi_\sigma$  for the rational map  $X \dashrightarrow Y$  induced by  $\sigma$  and  $R_{X,\sigma}$  for the closure of its ramification locus. Let  $\phi_{\sigma,b} : X_b \rightarrow \mathbb{P}^1$  be the restriction of  $\phi_\sigma$  to  $X_b$  and let  $R_{X,\sigma,b} = R_{X,\sigma} \cap X_b$  be the ramification locus of  $\phi_{\sigma,b}$ .

For  $L$  sufficiently ample and for each  $b \in B$ , we see by simple dimension counting that all of

$$(2.23) \quad \begin{aligned} &\{\sigma : \phi_\sigma(p_1) = \phi_\sigma(p_2) = \phi_\sigma(p_3) \\ &\quad \text{for three distinct points } p_1, p_2, p_3 \in D \cap X_b\}, \end{aligned}$$

$$(2.24) \quad \begin{aligned} &\{\sigma : \phi_\sigma(p_1) = \phi_\sigma(p_2) = \phi_\sigma(p_3) \\ &\quad \text{for } p_1 \neq p_2 \in D \cap X_b \text{ and } p_3 \in R_{X,\sigma,b}\}, \end{aligned}$$

$$(2.25) \quad \begin{aligned} &\{\sigma : \phi_\sigma(p_1) = \phi_\sigma(p_2) \text{ and } X_b \cap \text{Bs}(\sigma) \neq \emptyset, \\ &\quad \text{for } p_1 \neq p_2 \in D \cap X_b\} \end{aligned}$$

$$(2.26) \quad \begin{aligned} &\{\sigma : \phi_\sigma(p_1) = \phi_\sigma(p_2), \text{ where } p_1 \in D \cap X_b \text{ and} \\ &\quad \phi_{\sigma,b} \text{ ramifies at } p_2 \in R_{X,\sigma,b} \text{ with index } \geq 3\}, \end{aligned}$$

$$(2.27) \quad \begin{aligned} &\{\sigma : \phi_\sigma(p_1) = \phi_\sigma(p_2) \text{ and } X_b \cap \text{Bs}(\sigma) \neq \emptyset, \\ &\quad \text{where } p_1 \in D \cap X_b \text{ and } p_2 \in R_{X,\sigma,b}\}, \end{aligned}$$

$$(2.28) \quad \begin{aligned} &\{\sigma : \phi_\sigma(p_1) = \phi_\sigma(p_2), \text{ where } p_1 \neq p_2 \in R_{X,\sigma,b} \text{ and} \\ &\quad \phi_{\sigma,b} \text{ ramifies at } p_2 \text{ with index } \geq 3\}, \end{aligned}$$

$$(2.29) \quad \begin{aligned} &\{\sigma : \phi_\sigma(p_1) = \phi_\sigma(p_2) \text{ and } X_b \cap \text{Bs}(\sigma) \neq \emptyset, \\ &\quad \text{where } p_1 \neq p_2 \in R_{X,\sigma,b}\}, \end{aligned}$$

$$(2.30) \quad \{\sigma : \phi_{\sigma,b} \text{ ramifies at } p_1 \neq p_2 \in R_{X,\sigma,b} \text{ with indices } \geq 3\}$$

$$(2.31) \quad \begin{aligned} &\{\sigma : \phi_{\sigma,b} \text{ ramifies at } p_1 \in R_{X,\sigma,b} \text{ with index } \geq 3 \\ &\quad \text{and } X_b \cap \text{Bs}(\sigma) \neq \emptyset\} \text{ and} \end{aligned}$$

$$(2.32) \quad \{\sigma : \phi_{\sigma,b} \text{ ramifies at } p_1 \in R_{X,\sigma,b} \text{ with index } \geq 4\}$$

have codimension two in  $\mathbb{G}(1, |L|)$  and hence (2.22) follows. The same dimension count also shows that  $Y_b$  meets  $R$  transversely for  $b \in S$  and  $\sigma$  general. Hence if the equality in (2.22) holds,  $b \notin S$ .

Already by (2.22), we see that  $R$  has at worst double points as singularities. We can further show that the singularities  $R_{\text{sing}}$  of  $R$  are all nodes.

Let  $D = \sum D_i$ , where  $D_i$ 's are irreducible components of  $D$ , which are sections of  $X/B$  by our assumption on  $X$ . And let  $\Delta_{\sigma,i} = \phi_{\sigma}(D_i)$  and  $R_{Y,\sigma} = \phi_{\sigma}(R_{X,\sigma})$ . To show that  $R$  has normal crossing, it suffices to verify the following:

- $\Delta_{\sigma,i}$  and  $\Delta_{\sigma,j}$  meet transversely for all  $i \neq j$ ;
- $\Delta_{\sigma,i}$  meets  $R_{Y,\sigma}$  transversely for all  $i$ ;
- $R_{Y,\sigma}$  is nodal.

It is easy to see that the monodromy action on the intersections  $\Delta_{\sigma,i} \cap \Delta_{\sigma,j}$  when  $\sigma$  varies in  $\mathbb{G}(1, |L|)$  is transitive. Therefore, to show that  $\Delta_{\sigma,i}$  and  $\Delta_{\sigma,j}$  meet transversely, it suffices to show that they meet transversely at (at least) one point, i.e.,

- there exists  $\sigma \in \mathbb{G}(1, |L|)$ ,  $p_i \in D_i$  and  $p_j \in D_j$  such that  $\Delta_{\sigma,i}$  and  $\Delta_{\sigma,j}$  meet transversely at  $\phi_{\sigma}(p_i) = \phi_{\sigma}(p_j)$ .

Similarly, the other two statements translate to

- there exists  $\sigma \in \mathbb{G}(1, |L|)$ ,  $p_i \in D_i$  and  $q \in R_{X,\sigma}$  such that  $\Delta_{\sigma,i}$  and  $R_{Y,\sigma}$  meet transversely at  $\phi_{\sigma}(p_i) = \phi_{\sigma}(q)$ ;
- there exists  $\sigma \in \mathbb{G}(1, |L|)$  and  $q \in R_{X,\sigma,b}$  for some  $b$  such that  $\phi_{\sigma,b}$  has ramification index 3 at  $q$  and  $R_{Y,\sigma}$  is smooth at  $\phi_{\sigma}(q)$ ;
- there exists  $\sigma \in \mathbb{G}(1, |L|)$  and  $q_1 \neq q_2 \in R_{X,\sigma,b}$  for some  $b$  such that  $R_{Y,\sigma}$  has a node at  $\phi_{\sigma}(q_1) = \phi_{\sigma}(q_2)$ .

None of these statements are hard to prove. We leave their verification to the readers.  $\square$

Suppose that (1.4) fails for  $(X, D)$  and  $\{C_n \subset X\}$  is the sequence of irreducible curves satisfying (2.9). We fix a positive  $(1, 1)$  form  $\omega$  on  $X$  that represents  $c_1(L)$  and for every finite set of points  $S \subset B$ , we define

$$(2.33) \quad f_{\omega}(S) = \lim_{r \rightarrow 0} \varliminf_{n \rightarrow \infty} \left( \frac{1}{L \cdot C_n} \sum_{b \in S} \int_{C_n \cap \pi^{-1}(U(b,r))} \omega \right)$$

where  $U(b, r) \subset B$  is the disk of radius  $r$  centered at  $b$ . Of course, we need a metric on  $B$  in order to define  $U(b, r)$ . But it is obvious that the choice of metric on  $B$  is irrelevant here. Although  $f_{\omega}(S)$  depends on the choice of  $\omega$ , the vanishing of  $f_{\omega}(S)$  does not depend on  $\omega$ , i.e., if  $f_{\omega}(S) = 0$  for one  $\omega$ , it vanishes for all choices of  $\omega$ . And it is easy to see that

$$(2.34) \quad \sum_{\alpha} f_{\omega}(S_{\alpha}) \leq 1$$

for any collection  $\{S_{\alpha} \subset B\}$  of disjoint finite sets  $S_{\alpha}$ .

Let us fix a sufficient ample line bundle  $L$  on  $X$  and let  $\phi_\sigma : X \dashrightarrow Y$  be the map given by a pencil  $\sigma \subset |L|$  as in the proof of Proposition 2.5. This map gives rise to another log pair  $(Y, R)$  with  $R$  satisfying the conditions given in the above proposition. Let  $Q_\sigma \subset B$  be the finite set of points  $b$  where the equality in (2.22) holds. This gives us a map from  $\mathbb{G}(1, |L|)$  to  $B^N/S_N$  sending  $\sigma \rightarrow Q_\sigma$ , where  $N = |Q_\sigma|$  and  $B^N/S_N$  is the space of  $N$  unordered points on  $B$ . By Proposition 2.5,  $Q_\sigma \cap Q_{\sigma'} = \emptyset$  for two general pencils  $\sigma$  and  $\sigma'$ . Combining this with (2.34), we see that the set  $\{\sigma : f_\omega(Q_\sigma) > r\}$  is contained in a proper subvariety of  $\mathbb{G}(1, |L|)$  for every  $r > 0$ . Consequently, the set

$$(2.35) \quad \{\sigma : f_\omega(Q_\sigma) > 0\} = \bigcup_{n=1}^{\infty} \{\sigma : f_\omega(Q_\sigma) > \frac{1}{n}\}$$

is contained in a union of countably many proper subvarieties of  $\mathbb{G}(1, |L|)$ . In other words,  $f_\omega(Q_\sigma) = 0$  for a very general pencil  $\sigma$ . For a very general pencil  $\sigma$ ,  $C_n$  are disjoint from the base locus of  $\sigma$ . Hence  $L \cdot C_n = Y_p \cdot \Gamma_n$ , where  $\Gamma_n = \phi_\sigma(C_n)$  and  $Y_p$  is a fiber of  $Y/\mathbb{P}^1$ . In addition, we have proved that  $X_b \cap \text{Bs}(\sigma) = \emptyset$  for  $b \in Q_\sigma$ . Hence  $f_\omega(Q_\sigma) = 0$  implies

$$(2.36) \quad \lim_{r \rightarrow 0} \lim_{n \rightarrow \infty} \left( \frac{1}{Y_p \cdot \Gamma_n} \sum_{b \in Q_\sigma} \int_{\Gamma_n \cap \pi_Y^{-1}(U(b, r))} \eta \right) = 0$$

where  $\eta$  is the pullback of a positive  $(1, 1)$  form on  $\mathbb{P}^1$  representing  $c_1(\mathcal{O}_{\mathbb{P}^1}(1))$  and  $\pi_Y$  is the projection  $Y \rightarrow B$ . By taking a subsequence of  $\{\Gamma_n\}$ , we may as well replace  $\varprojlim$  by  $\lim$ .

We may further apply a suitable base change to  $Y/B$  to make  $R_Y$  into a union of sections of  $Y/B$  while preserving the other properties of  $(Y, R)$ . So we finally reduce the conjecture from  $(X, D, \varepsilon)$  to  $(Y, R, \delta)$ . Replacing  $(X, D, \varepsilon)$  by  $(Y, R, \delta)$ , we may assume the following holds.

**A1.**  $D \subset X = \mathbb{P}^1 \times B$  is a normal-crossing divisor which is a union of sections of  $X/B$ .

**A2.**  $\omega_{X/B} + D$  is relatively ample over  $B$ .

**A3.** For every fiber  $X_b$  of  $X/B$ ,

$$(2.37) \quad i_X(X_b, D) \geq X_b \cdot D - 1$$

**A4.** There is a sequence of reduced and irreducible curves  $\{C_n\}$  on  $X$  that dominate  $B$  and satisfy (2.9).

**A5.** Let  $Q \subset B$  be the set of points  $b$  where the equality in (2.37) holds, i.e.,  $Q = \pi(D_{\text{sing}})$ , where  $D_{\text{sing}}$  is the singular locus  $D_{\text{sing}}$  of  $D$ ; then

$$(2.38) \quad \lim_{r \rightarrow 0} \lim_{n \rightarrow \infty} \left( \frac{1}{X_p \cdot C_n} \sum_{b \in Q} \int_{C_n \cap \pi^{-1}(U(b, r))} w \right) = 0$$

where  $X_p$  is the fiber of  $X$  over a point  $p \in \mathbb{P}^1$  and  $w$  is the pullback of a positive  $(1, 1)$  form on  $\mathbb{P}^1$  representing  $c_1(\mathcal{O}_{\mathbb{P}^1}(1))$ .

## 3. PROOF OF CONJECTURE 1.3

**3.1. Lifts to the first jet space.** Now we can work exclusively on  $(X, D)$  with  $(X, D)$  satisfies the hypotheses A1-A5 in the last section. As in Vojta's proof of  $2 + \varepsilon$  theorem, we start by lifting every curve  $C_n \subset X$  to its 1st jet space.

Let  $\Omega_X(\log D)$  be the sheaf of logarithmic differentials with poles along  $D$  and  $T_X(-\log D) = \Omega_X(\log D)^\vee$  be its dual. Let  $Y = \mathbb{P}T_X(-\log D)$  be the projectivization of  $T_X(-\log D)$  with tautological line bundle  $L$ . Here we follow the traditional convention that

$$(3.1) \quad \mathbb{P}E = \text{Proj}(\oplus \text{Sym}^\bullet E^\vee) \text{ and } H^0(L) \cong H^0(E^\vee).$$

We have the basic exact sequence

$$(3.2) \quad 0 \rightarrow \pi^* \Omega_B \rightarrow \Omega_X(\log D) \rightarrow \Omega_{X/B}(D)$$

Note that this sequence is not right exact;  $\Omega_X(\log D) \rightarrow \Omega_{X/B}(D)$  fails to be surjective along  $D_{\text{sing}}$ .

Every nonconstant map  $\nu : C \rightarrow X$  from a smooth curve  $C$  to  $X$  can be naturally lifted to a map  $\nu_Y : C \rightarrow Y$  via the map

$$(3.3) \quad \mathbb{P}T_C(-\log \nu^* D) \rightarrow \mathbb{P}T_X(-\log D)$$

Suppose that  $\nu$  maps  $C$  birational onto its image. Then the natural map  $\nu^* \Omega_X(\log D) \rightarrow \Omega_C(\log \nu^* D)$  induces a map

$$(3.4) \quad \nu_Y^* L \rightarrow \Omega_C(\log \nu^* D)$$

Obviously, this map is nonzero and  $\nu_Y^* L$  is locally free; consequently, it is an injection. Therefore, we have

$$(3.5) \quad \deg \nu_Y^* L \leq \deg \Omega_C(\log \nu^* D) = 2g(C) - 2 + i_X(\nu(C), D)$$

Hence (1.4) holds if

$$(3.6) \quad \deg \nu_Y^* (\pi_X^* (\omega_{X/B} + D) - (1 + \varepsilon)L) \leq N_\varepsilon \deg(\nu^* X_b)$$

where  $\pi_X$  is the projection  $Y \rightarrow X$ . Another way to put this is that

$$(3.7) \quad G \cdot (\nu_Y)_* C \geq 0$$

for a sufficiently ample divisor  $M \subset B$  and every  $\nu : C \rightarrow X$  with  $\nu(C)$  dominating  $B$ , where

$$(3.8) \quad G = (1 + \varepsilon)L + \pi_B^* M - \pi_X^* (\omega_{X/B} + D)$$

where  $\pi_B = \pi \circ \pi_X$  is the projection  $Y \rightarrow B$ . Or in the context of our hypothesis A4, we want to show that

$$(3.9) \quad -G \cdot \Gamma_n = O(\deg C_n)$$

and thus arrive at a contradiction, where  $\Gamma_n \subset Y$  is the lift of  $C_n \subset X$  via its normalization and  $\deg C_n = C_n \cdot X_b$ . Here by  $O(\deg C_n)$ , we mean a quantity  $\leq K \deg C_n$  for some constant  $K$  and all  $n$ .

Obviously, (3.7) holds if the divisor  $G$  is numerically effective (NEF). Unfortunately, we cannot expect this to be true in general.



The map  $\Omega_X(\log D) \rightarrow \Omega_{X/B}(D)$  in (3.2) induces a rational map

$$(3.10) \quad \mathbb{P}T_{X/B}(-D) \dashrightarrow Y.$$

Let  $\Delta \subset Y$  be the closure of the image of this map. As we are going to see,  $\Delta$  will play a central role in our argument. Another way to characterize  $\Delta$  is the following.

**Lemma 3.1.** *We have*

$$(3.11) \quad \Delta = \overline{\bigcup_{b \in B} \mu_Y(X_b)}$$

and a curve  $\nu : C \hookrightarrow X$  is tangent to a fiber  $X_b$  if and only if  $\nu_Y(C)$  intersects  $\Delta$ , where  $\mu_Y : X_b \rightarrow Y$  is the lifting of the embedding  $X_b \hookrightarrow X$ .

*Proof.* This is more or less trivial.  $\square$

**3.2. Some Numerical Results.** Here we prove some numerical results on  $\Delta, L$  and  $G$ , which we are going to need later. First of all, it is obvious that  $\pi_X$  maps  $\Delta$  birationally one to  $X$ ; indeed, by a local analysis, we see that  $\Delta$  is the blowup of  $X$  along  $D_{\text{sing}}$ , i.e., the places where  $\Omega_X(\log D) \rightarrow \Omega_{X/B}(D)$  fails to be surjective. In the lift of  $\nu : C \rightarrow X$  to  $\nu_Y : C \rightarrow Y$ , if  $\nu$  is a smooth embedding, we have

$$(3.12) \quad (\nu_Y)^*L = \omega_C + \nu^{-1}(D)$$

where  $\nu^{-1}(D) = \text{supp}(\nu^*D)$  is the reduced pre-image of  $D$ . Namely, (3.4) is an isomorphism. Therefore, for every fiber  $X_b$ ,

$$(3.13) \quad L \cdot \tilde{X}_b = 2g(X_b) - 2 + i_X(X_b, D)$$

where  $\tilde{X}_b \subset \Delta$  is the proper transform of  $X_b$  under  $\Delta \rightarrow X$ . Applying this to all the fibers  $X_b$  with  $X_b \cap D_{\text{sing}} \neq \emptyset$ , we see that

$$(3.14) \quad L|_{\Delta} = \pi_X^*(\omega_{X/B} + D + \pi^*M) - E$$

for some divisor  $M$  on  $B$ , where

$$(3.15) \quad E = \sum_{q \in D_{\text{sing}}} E_q$$

is the exceptional divisor of  $\Delta \rightarrow X$ . To determine  $M$ , we restrict everything to a section  $X_p = \rho^{-1}(p)$  of  $X/B$ , where  $\rho$  is the projection  $X \rightarrow \mathbb{P}^1$ . For  $p$  general, the restriction of (3.2) to  $X_p \cong B$  becomes

$$(3.16) \quad 0 \rightarrow \Omega_{X_p} \rightarrow \Omega_X(\log D)|_{X_p} \rightarrow \mathcal{O}_{X_p}(D) \rightarrow 0$$

Let  $\Delta_p$  be the proper transform of  $X_p$  under  $\Delta \rightarrow X$ . Then we see from (3.16) that the restriction of  $L$  to  $\Delta_p \cong B$  is

$$(3.17) \quad L|_{\Delta_p} = \pi_X^*D$$

Comparing (3.14) and (3.17), we conclude that  $M$  is trivial and hence

$$(3.18) \quad L|_{\Delta} = \pi_X^*(\omega_{X/B} + D) - E$$

As a consequence,

$$(3.19) \quad \begin{aligned} G|_{\Delta} &= ((1 + \varepsilon)L + \pi_B^*M - \pi_X^*(\omega_{X/B} + D))|_{\Delta} \\ &= \varepsilon\pi_X^*(\omega_{X/B} + D) + \pi_B^*M - (1 + \varepsilon)E \end{aligned}$$

Next, we claim that

$$(3.20) \quad \Delta = L - \pi_B^*\omega_B$$

This is obviously true if (3.2) is an exact sequence of locally free sheaves, i.e.,  $D_{\text{sing}} = \emptyset$ . To see this is true in general, we restrict everything to a smooth curve  $C \subset X$  with  $C \cap D_{\text{sing}} = \emptyset$ . By the above reason, (3.20) holds when restricted to  $\pi_X^{-1}(C)$ . Such curves  $C$  obviously generate  $\text{Pic}(X)$  and hence (3.20) holds over  $Y$ .

By restricting (3.2) to each fiber  $X_b$  of  $X/B$ , we see that  $L$  is relatively NEF over  $B$ . Moreover, the following holds.

**Lemma 3.2.** *For all  $m \geq k \in \mathbb{Z}$ ,  $mL - k\Delta$  is relatively BPF over  $B$  and*

$$(3.21) \quad H^1(m(L + \pi_B^*M) - k\Delta) = 0$$

for a sufficiently ample divisor  $M \subset B$ .

*Proof.* Since  $c_1(\Omega_X(\log D)) = \omega_X + D$ , the restriction of  $\Omega_X(\log D)$  to a fiber  $X_b \cong \mathbb{P}^1$  is

$$(3.22) \quad \Omega_X(\log D)|_{X_b} = \mathcal{O}_{\mathbb{P}^1}(\beta) \oplus \mathcal{O}_{\mathbb{P}^1}(\gamma)$$

where

$$(3.23) \quad \beta + \gamma = \alpha = (\omega_{X/B} + D) \cdot X_b.$$

By (3.2), we must have  $\beta, \gamma \geq 0$ . Therefore,

$$(3.24) \quad Y_b \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-\beta) \oplus \mathcal{O}_{\mathbb{P}^1}(-\gamma))$$

and together with (3.20), we see that  $mL - k\Delta$  is relatively NEF over  $B$  for  $m \geq k$ . Also we see from the above argument that

$$(3.25) \quad H^1(Y_b, mL - k\Delta) = 0 \Leftrightarrow R^1(\pi_B)_*\mathcal{O}(mL - k\Delta) = 0$$

This implies

$$(3.26) \quad \begin{aligned} H^1(m(L + \pi_B^*M) - k\Delta) &= H^1((\pi_B)_*\mathcal{O}(m(L + \pi_B^*M) - k\Delta)) \\ &= H^1((\pi_B)_*L^{m-k} \otimes \mathcal{O}_B(k\omega_B + mM)) \end{aligned}$$

By (3.24),

$$(3.27) \quad \text{Sym}^n H^0(Y_b, L) = H^0(Y_b, L^n).$$

Therefore,

$$(3.28) \quad H^1(m(L + \pi_B^*M) - k\Delta) = H^1(\text{Sym}^{m-k}(\pi_B)_*L \otimes \mathcal{O}_B(k\omega_B + mM))$$

It suffices to choose  $M$  such that all of  $M$ ,  $\omega_B + M$  and  $p_*L \otimes \mathcal{O}_B(M)$  are ample and (3.21) follows.  $\square$

*Remark 3.3.* It is possible to give a more precise version of (3.21) on how ample  $M$  should be in terms of  $\omega_B$  and  $D$ ; however, we have no need for it here. Also in the above proof, we observe that  $L$  fails to be ample on  $Y_b$  if and only if (3.22) splits as

$$(3.29) \quad \Omega_X(\log D)|_{X_b} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(\alpha)$$

If (3.29) holds on a general fiber  $X_b$ , it holds everywhere and this only happens when  $D$  consists of  $\alpha + 2$  disjoint sections of  $X/B$ , in which case the conjecture is trivial. Hence we may assume that  $L$  is ample on a general fiber of  $Y/B$ . This implies that  $L + \pi_B^* M$  is big for a sufficiently ample divisor  $M \subset B$ , in addition to being NEF as already proved. The same, of course, holds for  $mL - k\Delta + \pi_B^* M$  when  $m > k$ .

**3.3. Bergman Metric.** Given a line bundle  $L$  on a compact complex manifold  $X$  and sections  $s_0, s_1, \dots, s_n \in |L|$  of  $L$ , we recall the *Bergman metric* associated to  $\{s_k\}$  is the pullback of the Fubini-Study metric under the map  $X \dashrightarrow \mathbb{P}^n$  given by  $\{s_k\}$ , i.e., the pseudo-metric with associated  $(1, 1)$  form

$$(3.30) \quad w = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left( \sum_{k=0}^n |s_k|^2 \right)$$

Alternatively, Fubini-Study metric can be regarded as a metric of the line bundle  $\mathcal{O}_{\mathbb{P}^n}(1)$  and the Bergman metric is correspondingly a pseudo-metric of  $L$  with  $w$  the curvature form. In general,  $w$  is only a closed real current of type  $(1, 1)$  with the following properties:

- it is  $C^\infty$  outside of the base locus  $\text{Bs}\{s_k\}$  of  $\{s_k\}$ ;
- it represents  $c_1(L)$  if  $\{s_k\}$  is BPF;
- we always have

$$(3.31) \quad \nu^* w \text{ is } C^\infty, \nu^* w \geq 0 \text{ and } \deg(\nu^* L) \geq \int_C \nu^* w$$

for any morphism  $\nu : C \rightarrow X$  from a smooth and irreducible projective curve  $C$  to  $X$  with  $\nu(C) \not\subset \text{Bs}\{s_k\}$ .

The indeterminacy of the rational map  $\phi : X \dashrightarrow \mathbb{P}^n$  given by  $\{s_k\}$  can be resolved by a sequence of blowups along smooth centers over  $\text{Bs}\{s_k\}$ . That is, there exists a birational map  $\pi : Y \rightarrow X$  such that  $f = \phi \circ \pi$  is regular. Let  $\tilde{s}_k$  be the proper transform of  $s_k$  under  $\pi$ . Then  $\{\tilde{s}_k\}$  span a BPF linear system of  $\tilde{L} = f^* \mathcal{O}_{\mathbb{P}^n}(1)$ . Let  $\tilde{w}$  be the Bergman metric associated to  $\{\tilde{s}_k\}$ . Then  $\tilde{w} = \pi^* w$  outside of exceptional locus of  $\pi$ . Indeed, the current  $w$  is defined in the way of

$$(3.32) \quad \langle w, \gamma \rangle = \int_Y \tilde{w} \wedge \pi^* \gamma$$

Then (3.31) follows easily.

**3.4. Construction of the 1st chern classes.** Let

$$(3.33) \quad \pi_X^*(\omega_{X/B} + D) = \alpha Y_p + \pi_B^* N$$

for some divisor  $N \subset B$ , where  $Y_p$  is a fiber of  $Y/\mathbb{P}^1$ . We replace  $M$  by  $M + N$  and write  $G$  in the form

$$(3.34) \quad G = (1 + \varepsilon)L + \pi_B^* M - \alpha Y_p$$

Our purpose remains, of course, to show (3.9).

We write the LHS of (3.9) in the integral form:

$$(3.35) \quad G \cdot \Gamma_n = \int_{\Gamma_n} c_1(G) = \int_{\Gamma_n \setminus U} c_1(G) + \int_{\Gamma_n \cap U} c_1(G)$$

where  $U$  is an (analytic) open neighborhood of  $\Delta$ . Here we have to work with the forms that represent the first chern classes instead of cohomology classes themselves, i.e.,  $c_1(G)$  refers to a  $(1,1)$  form representing the first chern class of  $G$ ; otherwise, the integrals in (3.35) do not make sense. The construction of appropriate  $c_1(G)$  is one of the main parts of our proof. Basically, by a proper choice of  $c_1(G)$  with

$$(3.36) \quad c_1(G) = c_1((1 + \varepsilon)L + \pi_B^* M) - c_1(\alpha Y_p)$$

we will show that both

$$(3.37) \quad - \int_{\Gamma_n \setminus U} c_1(G) \text{ and } - \int_{\Gamma_n \cap U} c_1(G)$$

have order of  $O(\deg C_n)$ . The forms representing  $c_1((1 + \varepsilon)L + \pi_B^* M)$  and  $c_1(\alpha Y_p)$  are constructed via Bergman metric mentioned above.

Let us first fix a sufficiently large integer  $m$  with  $m\varepsilon \in \mathbb{Z}$ ; obviously, we may assume  $\varepsilon \in \mathbb{Q}$ . Since  $H^0(m\alpha Y_p) = H^0(\mathcal{O}_{\mathbb{P}^1}(m\alpha))$ , a general pencil of  $m\alpha Y_p$  is BPF. To construct a form  $w$  representing  $c_1(m\alpha Y_p)$ , it is enough to choose a BPF pencil of  $m\alpha Y_p$  with basis  $\{s_0, s_1\}$  and let

$$(3.38) \quad w = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log(|s_0|^2 + |s_1|^2)$$

be the Bergman metric associated to  $\{s_0, s_1\}$ . Obviously,  $w$  is  $C^\infty$  and represents  $c_1(m\alpha Y_p)$ . Next we will construct a Bergman metric on the line bundle  $\mathcal{O}_Y(m(1 + \varepsilon)L + m\pi_B^* M)$ .

Let  $S_i = \{s_i = 0\}$  for  $i = 0, 1$  and let  $\{\sigma_{0j} : j \in J\}$  be a basis of the linear system of  $m(1 + \varepsilon)L + m\pi_B^* M$  consisting of sections  $\sigma$  with

$$(3.39) \quad \sigma|_{S_0} \in H^0(S_0, m(1 + \varepsilon)L + m\pi_B^* M - 2\Delta)$$

Or equivalently,  $\sigma_{0j}$  are the sections tangent to  $S_0$  along  $S_0 \cap \Delta$ .

**Lemma 3.4.** *For each  $j$ , there exists a section  $\sigma_{1j}$  of  $m(1 + \varepsilon)L + m\pi_B^* M$  such that  $s_0\sigma_{1j} - s_1\sigma_{0j}$  vanishes to the order of 2 along  $\Delta$ , i.e.,*

$$(3.40) \quad s_0\sigma_{1j} - s_1\sigma_{0j} \in H^0(m(1 + \varepsilon)L + m\pi_B^* M + m\alpha Y_p - 2\Delta)$$

In addition,  $\{\sigma_{1j}\}$  can be chosen to be a basis of the linear system consisting of sections  $\sigma$  with

$$(3.41) \quad \sigma|_{S_1} \in H^0(S_1, m(1 + \varepsilon)L + m\pi_B^*M - 2\Delta)$$

*Proof.* Let  $F_0$  be the subscheme of  $Y$  given by  $F_0 = S_0 \cap 2\Delta$ . Then we have the Koszul complex for the ideal sheaf  $I_{F_0}$  of  $F_0 \subset Y$ :

$$(3.42) \quad 0 \rightarrow \mathcal{O}(-S_0 - 2\Delta) \rightarrow \mathcal{O}(-S_0) \oplus \mathcal{O}(-2\Delta) \rightarrow I_{F_0} \rightarrow 0$$

Obviously,

$$(3.43) \quad \Sigma_0 = H^0(\mathcal{O}_Y(m(1 + \varepsilon)L + m\pi_B^*M) \otimes I_{F_0})$$

is exactly the linear system  $\text{Span}\{\sigma_{0j}\}$  generated by  $\{\sigma_{0j}\}$ . By Lemma 3.2,

$$(3.44) \quad \begin{aligned} & H^1(m(1 + \varepsilon)L + m\pi_B^*M + m\alpha Y_p - S_0 - 2\Delta) \\ &= H^1(m(1 + \varepsilon)L + m\pi_B^*M - 2\Delta) = 0 \end{aligned}$$

Therefore,  $AF + BG$  holds for

$$(3.45) \quad s_1\sigma_{0j} \in H^0(\mathcal{O}_Y(m(1 + \varepsilon)L + m\pi_B^*M + m\alpha Y_p) \otimes I_{F_0})$$

That is,

$$(3.46) \quad s_1\sigma_{0j} = s_0\sigma_{1j} + s_\Delta^2 l_j$$

for some  $\sigma_{1j}$ , where  $\Delta = \{s_\Delta = 0\}$  and  $l_j$  is a section of

$$(3.47) \quad \mathcal{O}_Y(m(1 + \varepsilon)L + m\pi_B^*M + m\alpha Y_p - 2\Delta)$$

And (3.40) follows. Obviously,  $\sigma_{1j}$  are members of the linear sytem

$$(3.48) \quad \Sigma_1 = H^0(\mathcal{O}_Y(m(1 + \varepsilon)L + m\pi_B^*M) \otimes I_{F_1})$$

where  $I_{F_1}$  is the ideal sheaf of the subscheme  $F_1 = S_1 \cap 2\Delta \subset Y$ . It is obvious that

$$(3.49) \quad H^0(m(1 + \varepsilon)L + m\pi_B^*M - 2\Delta) \subset \Sigma_0 \cap \Sigma_1$$

It is not hard to see that  $\{\sigma_{1j}\}$  spans the quotient

$$(3.50) \quad \Sigma_1 / H^0(m(1 + \varepsilon)L + m\pi_B^*M - 2\Delta) = \text{Span}\{\sigma_{1j}\}$$

Without loss of generality, we may assume that  $\{\sigma_{0j} : j \in J\}$  contains a subset  $\{\sigma_{0j} : j \in J_\Delta\}$  which is a basis of  $H^0(m(1 + \varepsilon)L + m\pi_B^*M - 2\Delta)$ , where  $J_\Delta \subset J$ . Then it is enough to choose  $\sigma_{1j} = \sigma_{0j}$  for  $j \in J_\Delta$ . Combining this with (3.50), we see that  $\{\sigma_{1j}\}$  is a basis of  $\Sigma_1$ .  $\square$

Let  $\sigma_{1j}$  be the sections given in the above lemma. Together with  $\{\sigma_{0j}\}$  we have the Bergman metric associated to  $\{\sigma_{ij} : 0 \leq i \leq 1, j \in J\}$

$$(3.51) \quad \gamma = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left( \sum_{ij} |\sigma_{ij}|^2 \right)$$

And we let

$$(3.52) \quad \eta = \gamma - w$$

**Proposition 3.5.** *Let  $\Sigma_i$  be the linear system generated by  $\{\sigma_{ij} : j \in J\}$  as in (3.43) and (3.48). For each  $i$ , the base locus of  $\Sigma_i$  is contained in  $(Y_Q \cup S_i) \cap \Delta$ , where  $Q = \pi(D_{\text{sing}}) \subset B$  is the finite set defined in A5 and  $Y_Q = \pi_B^{-1}(Q)$ .*

*Proof.* Since  $H^0(m(1+\varepsilon)L + m\pi_B^*M - 2\Delta) \subset \Sigma_i$ , the base locus  $\text{Bs}(\Sigma_i)$  of  $\Sigma_i$  is contained in  $\Delta$  by Lemma 3.2. So it suffices to show that  $\text{Bs}(\Sigma_i) \subset Y_Q \cup S_i$ .

Let  $F_i = S_i \cap 2\Delta$  be the subscheme of  $Y$  defined in the proof of Lemma 3.4. We have the exact sequence

$$\begin{aligned}
 (3.53) \quad & 0 \rightarrow \mathcal{O}_Y(m(1+\varepsilon)L + m\pi_B^*M - 2\Delta) \\
 & \rightarrow \mathcal{O}_Y(m(1+\varepsilon)L + m\pi_B^*M) \otimes I_{F_i} \\
 & \rightarrow \underbrace{\mathcal{O}_\Delta(m(1+\varepsilon)L + m\pi_B^*M - S_i)}_{\mathcal{O}_\Delta(mG)} \otimes \mathcal{O}_Y/I_\Delta^2 \rightarrow 0
 \end{aligned}$$

where  $I_\Delta = \mathcal{O}_Y(-\Delta)$  is the ideal sheaf of  $\Delta \subset Y$ . Again by Lemma 3.2,  $H^1(\mathcal{O}_Y(m(1+\varepsilon)L + m\pi_B^*M - 2\Delta)) = 0$  and hence we have the surjection

$$\begin{aligned}
 (3.54) \quad & \Sigma_i \twoheadrightarrow H^0(\mathcal{O}_\Delta(m(1+\varepsilon)L + m\pi_B^*M - S_i) \otimes \mathcal{O}_Y/I_\Delta^2) \\
 & \cong H^0(\mathcal{O}_\Delta(mG) \otimes \mathcal{O}_Y/I_\Delta^2)
 \end{aligned}$$

Composing the above map with

$$\begin{aligned}
 (3.55) \quad & \varphi : H^0(\mathcal{O}_\Delta(mG) \otimes \mathcal{O}_Y/I_\Delta^2) \rightarrow H^0(\mathcal{O}_\Delta(mG) \otimes \mathcal{O}_Y/I_\Delta) \\
 & = H^0(\mathcal{O}_\Delta(mG))
 \end{aligned}$$

we have a natural map

$$(3.56) \quad f : \Sigma_i \rightarrow H^0(\mathcal{O}_\Delta(mG)).$$

To show that  $\text{Bs}(\Sigma_i) \subset Y_Q \cup S_i$ , it is enough to show that

$$(3.57) \quad \text{Bs}(f(\Sigma_i)) \subset Y_Q$$

which is equivalent to

$$(3.58) \quad \text{Bs}(\text{Im}(\varphi)) \subset Y_Q$$

by (3.54). For  $M \subset B$  sufficiently ample, we have the diagram

$$\begin{array}{ccc}
 (3.59) \quad & H^0(\mathcal{O}_\Delta(mG) \otimes \mathcal{O}_Y/I_\Delta^2) & \twoheadrightarrow H^0(\Delta_b, \mathcal{O}_\Delta(mG) \otimes \mathcal{O}_Y/I_\Delta^2) \\
 & \downarrow \varphi & \downarrow \varphi_b \\
 & H^0(\mathcal{O}_\Delta(mG)) & \twoheadrightarrow H^0(\Delta_b, \mathcal{O}_\Delta(mG))
 \end{array}$$

with rows being surjections when we restrict  $\varphi$  to each fiber  $\Delta_b$  of  $\Delta/B$ . Therefore, it suffices to show that

$$(3.60) \quad \text{Bs}(\text{Im}(\varphi_b)) = \emptyset$$

for all  $b \notin Q$ . This is more or less obvious since we have the exact sequence

$$(3.61) \quad \begin{array}{ccccccc} 0 & \longrightarrow & I_\Delta/I_\Delta^2 & \longrightarrow & \mathcal{O}_Y/I_\Delta^2 & \longrightarrow & \mathcal{O}_Y/I_\Delta \longrightarrow 0 \\ & & \parallel & & & & \parallel \\ & & \mathcal{O}_\Delta(-\Delta) & & & & \mathcal{O}_\Delta \end{array}$$

When we tensor the sequence by  $\mathcal{O}_\Delta(mG)$  and restrict it to  $\Delta_b \cong \mathbb{P}^1$  with  $b \notin Q$ , we have

$$(3.62) \quad h^1(\Delta_b, \mathcal{O}_\Delta(mG - \Delta)) = h^1(\mathcal{O}_{\mathbb{P}^1}((m\varepsilon - 1)\alpha)) = 0$$

by (3.19) and (3.20). Consequently,  $\varphi_b$  is surjective and

$$(3.63) \quad \text{Bs}(\text{Im}(\varphi_b)) = \text{Bs}(H^0(\Delta_b, \mathcal{O}_\Delta(mG))) = \text{Bs}(H^0(\mathcal{O}_{\mathbb{P}^1}(m\varepsilon\alpha))) = \emptyset$$

□

*Remark 3.6.* It is not hard to see that the above proposition continues to hold with tangency 2 replaced by any  $\mu \leq m\varepsilon$ . Moreover, being a little more careful, we can actually show that

$$(3.64) \quad \text{Bs}(\Sigma_i) = \tilde{X}_Q \cup (S_i \cap \Delta)$$

where  $\tilde{X}_Q \subset \Delta$  is the proper transform of  $X_Q = \pi^{-1}(Q)$  under the map  $\Delta \rightarrow X$ . However, we have no need for these here.

By the above proposition, we see that the base locus of  $\{\sigma_{ij} : i, j\}$  is supported on  $Y_Q \cap \Delta$ . Consequently,  $\gamma$  is a closed  $(1, 1)$  current which is  $C^\infty$  on  $Y \setminus (Y_Q \cap \Delta)$ . By (3.31),

$$(3.65) \quad \begin{aligned} -mG \cdot \Gamma_n &\leq -\int_{\Gamma_n} \eta = -\int_{\Gamma_n \setminus U} \eta - \int_{\Gamma_n \cap U} \eta \\ &\leq \int_{\Gamma_n \setminus U} w - \int_{\Gamma_n \cap U} \eta \end{aligned}$$

The fact that the first integral has order of  $O(\deg C_n)$  is a consequence of the following lemma.

**Lemma 3.7.** *Let  $U \subset Y$  be an open neighborhood of  $\Delta$ ,  $w$  be a smooth  $(1, 1)$  form on  $X$  and  $\kappa$  be a positive smooth  $(1, 1)$  form on  $B$ . Then there exists a constant  $A_U > 0$  such that at every point  $(p, v) \in Y \setminus U$*

$$(3.66) \quad |\langle w, v \wedge \bar{v} \rangle| \leq A_U \langle \pi^* \kappa, v \wedge \bar{v} \rangle$$

where  $p \in X$  and  $v \in T_{X,p}(-\log D)$ .

*Proof.* By Lemma 3.1,  $\langle \pi^* \kappa, v \wedge \bar{v} \rangle$  does not vanish for  $(p, v) \notin \Delta$  and hence the function

$$(3.67) \quad f(p, v) = \frac{\langle w, v \wedge \bar{v} \rangle}{\langle \pi^* \kappa, v \wedge \bar{v} \rangle}$$

is continuous on  $Y \setminus \Delta$ . Then (3.66) follows from the compactness of  $Y \setminus U$ . □

Note that  $w$  is the pullback of a form on  $X$ ; indeed, it is the pullback of a form on  $\mathbb{P}^1$ . So Lemma 3.7 applies and we conclude that

$$(3.68) \quad w \leq A_U \pi_B^* \kappa$$

on  $\Gamma_n \setminus U$  for some constant  $A_U$  depending only on  $U$ , where we choose  $\kappa$  to be a positive  $(1, 1)$  form on  $B$  representing  $c_1(\mathcal{O}_B(b))$  for a point  $b \in B$ . Therefore,

$$(3.69) \quad \int_{\Gamma_n \setminus U} w \leq A_U \int_{\Gamma_n} \pi_B^* \kappa = A_U \deg(C_n)$$

Next, we claim that  $\eta > 0$  everywhere on  $\Delta \setminus Y_Q$ .

**Lemma 3.8.** *The current  $\eta > 0$  at every point  $p \in \Delta \setminus Y_Q$ .*

By (2.38), there exists an open neighborhood  $V$  of  $Y_Q$  such that

$$(3.70) \quad \int_{\Gamma_n \cap V} w \leq \varepsilon(m\alpha Y_p \cdot \Gamma_n)$$

By the above lemma and the compactness of  $\Delta \setminus V$ , we see that  $\eta > 0$  in  $U \setminus V$  for some open neighborhood  $U$  of  $\Delta$ . The second integral in (3.65) becomes

$$(3.71) \quad \begin{aligned} - \int_{\Gamma_n \cap U} \eta &\leq - \int_{\Gamma_n \cap (U \setminus V)} \eta + \int_{\Gamma_n \cap V} w \\ &\leq \varepsilon(m\alpha Y_p \cdot \Gamma_n) = m\varepsilon(\omega_{X/B} + D) \cdot C_n + O(\deg C_n) \end{aligned}$$

Combining (3.69) and (3.71), we have

$$(3.72) \quad \begin{aligned} -G \cdot \Gamma_n &= \varepsilon(\omega_{X/B} + D) \cdot C_n + O(\deg C_n) \\ \Rightarrow -((1 + \varepsilon)L + \pi_B^* M - (1 - \varepsilon)\pi_X^*(\omega_{X/B} + D)) \cdot \Gamma_n &= O(\deg C_n) \end{aligned}$$

Replace  $\varepsilon$  by  $\varepsilon/(2 + \varepsilon)$  and we are done. It remains to verify Lemma 3.8.

*Proof of Lemma 3.8.* At least one of  $s_0(p)$  and  $s_1(p)$  does not vanish. Let us assume that  $s_0(p) \neq 0$  WLOG. Let  $r_j = \sigma_{0j}/s_0$ ;  $r_j$  is holomorphic at  $p$ , of course. Let  $\delta_j = \sigma_{1j} - s_1 r_j$ . By our construction of  $\sigma_{1j}$ , we see that  $\delta_j$  vanishes to the order of 2 along  $\Delta$ . We may write

$$(3.73) \quad \begin{aligned} \gamma &= \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left( \sum_j (|s_0 r_j|^2 + |s_1 r_j + \delta_j|^2) \right) \\ &= \underbrace{\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log(|s_0|^2 + |s_1|^2)}_w + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left( \sum_j |r_j|^2 \right) \\ &\quad - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left( 1 + \sum_j \frac{s_1 r_j \bar{\delta}_j + \bar{s}_1 \bar{r}_j \delta_j + |\delta_j|^2}{(|s_0|^2 + |s_1|^2) \sum_j |r_j|^2} \right) \end{aligned}$$



Basically, we want to show that the last term in (3.73) vanishes along  $\Delta$ . Then

$$(3.74) \quad \eta|_{\Delta} = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left( \sum_j |r_j|^2 \right)$$

locally at  $p$ , which is positive.

Since  $\eta$  is  $C^\infty$  at  $p$ , it is enough to show that  $\eta > 0$  at  $p$  when  $\eta$  is restricted to every curve passing through  $p$ , i.e., to show that  $f^*\eta > 0$  at  $q$  for every nonconstant morphism  $f : C \rightarrow Y$  from a smooth and irreducible projective  $C$  to  $Y$  with  $f(q) = p$ . Indeed, it is enough to show the following

for every tangent vector  $\xi \in T_{Y,p}$ , there exists a morphism  $f : C \rightarrow Y$  from a smooth irreducible curve  $C$  to  $Y$  with  $f(q) = p$ ,  $\xi \in f_*T_{C,q}$  and  $f^*\eta > 0$  at  $q$ .

Therefore, we can also exclude the curves contained in a fixed proper subvariety of  $Y$ . So we may assume that  $f(C) \not\subset \Delta \cup W$ , where  $W \subsetneq Y$  is the subvariety such that

$$(3.75) \quad L \cdot \Gamma = 0 \text{ for a curve } \Gamma \Leftrightarrow \Gamma \subset W$$

Such  $W$  exists because  $L$  is big and NEF (see Remark 3.3). Let

$$(3.76) \quad \hat{\mathcal{O}}_{C,q} \cong \mathbb{C}[[t]]$$

be the formal local ring of  $C$  at  $q$  and  $\mu$  be its valuation, i.e.,  $\mu(t^n) = n$ . Let

$$(3.77) \quad \mu(f^*s_\Delta) = \lambda$$

where  $\Delta = \{s_\Delta = 0\}$ . Then  $\mu(f^*\delta_j) \geq 2\lambda$ . And since  $\{\sigma_{0j}\}$  and hence  $\{r_j\}$  are BPF at  $p$ , we have

$$(3.78) \quad f^* \left( \frac{s_1 r_j \bar{\delta}_j + \bar{s}_1 \bar{r}_j \delta_j + |\delta_j|^2}{(|s_0|^2 + |s_1|^2) \sum_j |r_j|^2} \right) = O(t^{2\lambda} + \bar{t}^{2\lambda} + |t|^{4\lambda})$$

Therefore, we obtain

$$(3.79) \quad \frac{\sqrt{-1}}{2\pi} f^* \partial \bar{\partial} \log \left( 1 + \sum_j \frac{s_1 r_j \bar{\delta}_j + \bar{s}_1 \bar{r}_j \delta_j + |\delta_j|^2}{(|s_0|^2 + |s_1|^2) \sum_j |r_j|^2} \right) \Big|_{t=0} = 0$$

by the Taylor expansion of the LHS. Consequently,

$$(3.80) \quad \begin{aligned} f^*\eta|_q &= \frac{\sqrt{-1}}{2\pi} f^* \partial \bar{\partial} \log \left( \sum_j |r_j|^2 \right) \Big|_q \\ &= \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left( \sum_j |f^*\sigma_{0j}|^2 \right) \Big|_q \end{aligned}$$

Since  $H^0(m(1 + \varepsilon)L + m\pi_B^*M - 2\Delta) \subset \Sigma_0$  and  $f(C) \not\subset \Delta \cup W$ , the linear system  $f^*\Sigma_0$  is big on  $C$ . Therefore,  $f^*\eta > 0$  at  $q$  and  $\eta > 0$  at  $p$ .  $\square$

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